JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 19, No. 5, September–October 1996

Analytical Vibration and Resonant Motion of a Stretched Spinning Nonlinear Tether

Albert C. J. Luo,* Ray P. S. Han,† and G. Tyc‡
University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada
V. J. Modi§
University of British Columbia, Vancouver, British Columbia V6T 1Z4, Canada
and
A. K. Misra¶
McGill University, Montreal, Quebec H3A 2K6, Canada

A nonlinear mechanical model of a spinning tether in three-dimensional space is constructed to study its resonant motion and stability characteristics. The model is reduced via the Galerkin method to linear and nonlinear coupled systems. For the linear case, exact solutions as well as resonant and stability conditions for both free and forced vibrations of the undamped and damped tether are obtained. Also given are the conditions pertaining to the occurence of internal, external, and subharmonic resonance. For the nonlinear system, exact solutions for the free vibration as well as the resonant conditions for the free and forced vibrations of the undamped tether are derived. Several numerical examples illustrating the resonant motion are presented.

I. Introduction

MANKIND has been fascinated by the vibration of strings since the invention of their use in musical instruments, and in recent years, there has been an increasing worldwide interest in the use of tethers in space. In this paper, we are interested in the linear and nonlinear motion of a spinning tether. Our motivation for this study comes from our involvement in the Tether Dynamics Experiment on the Canadian Space Agency's OEDIPUS-C sounding-rocket mission,[1] which was launched on Nov. 6, 1995. This mission, together with an earlier mission flown in 1989 (OEDIPUS-A; Ref. 2), employed a unique configuration consisting of two spinning payloads connected by a 1-km long tether. The entire system is spin-stabilized about its longitudinal axis. One characteristic that differentiates the OEDIPUS missions from all other known space tether missions is that they employ a spinning tether. It is this spin about the tether's longitudinal axis that complicates the ensuing dynamics and stability of the motion. This is particularly true when the motion of the tether involves large deflections.

Carrier[3] was the first to study the large-amplitude free vibration of a string using a perturbation technique. In the 1960s, stretched string models with large-amplitude nonplanar motion were described by several researchers.[4–7] Methods for handling the stability of nonlinear vibrations in physical systems have been presented by Hayashi.[8] The equations for large-vibrations of strings have been revisited by Antman[9] and Gough.[10] More recently, bifurcation and chaotic motion of nonlinear strings have stirred considerable interest in engineers and scientists, and valuable results were contained in several publications.[11–17] The foregoing pertains to a nonspinning tether, and to the best of our knowledge, a nonlinear dynamic analysis of a spinning tether has not been attempted.

The three-dimensional nonlinear equations for the vibration of a damped stretched spinning tether are presented. The resulting equations are converted into ordinary differential equations via the Galerkin method. Exact solutions for the free and forced vibrations and the stability and resonant conditions are derived, for both undamped and damped linear spinning tethers.[18] From the undamped results, the internal and external resonant conditions of the spinning tether are deduced; the subharmonic resonant conditions are also obtained. It is shown that external excitation resonance occurs when there are two or more external excitations acting on the spinning tether. Results confirm that although external damping always stabilizes the motion of the spinning tether, this is not necessarily true with internal damping. For the nonlinear spinning tether, exact solutions for the free vibration of the undamped tether are derived.[19] From these nonlinear solutions and employing an energy analysis of the system, the resonant conditions are obtained.

II. Mechanical Model

Figure 1 depicts a spinning tether stretched along the positive x axis of a rectangular Cartesian coordinate system such that one of its ends is located at the origin o. Let l denote the length of the tether at rest and Ω the angular frequency of the stretched tether. Note that all points on the tether are defined with respect to its at-rest configuration. Denoting the displacements parallel to the x, y, and z axes by u, v, and w, respectively, the equations of motion for a stretched spinning tether are

$$u_{tt} + \delta_e u_t - \delta_{if} u_{xxt} - \frac{\partial}{\partial x}\left\{\frac{c_2^2 + c_1^2[1 + \delta_{im}(\mathrm{d}/\mathrm{d}t)]\left[\sqrt{(1+u_x)^2 + v_x^2 + w_x^2} - 1\right]}{\sqrt{(1+u_x)^2 + v_x^2 + w_x^2}}(1+u_x)\right\} = \bar{q}_x \tag{1}$$

Received June 15, 1995; revision received Feb. 22, 1996; accepted for publication April 14, 1996.

*Postdoctoral Fellow, Department of Mechanical and Industrial Engineering.

†Professor, Department of Mechanical and Industrial Engineering; currently Professor, Department of Mechanical Engineering, University of Iowa, Iowa City, IA 52242-1527. Senior Member AIAA.

‡Ph.D. Student, Department of Mechanical and Industrial Engineering; also Mechanical Systems Engineer, Space Business Group, Bristol Aerospace Limited, Winnipeg, Manitoba R3C 2S4, Canada. Member AIAA.

§Professor, Department of Mechanical Engineering. Fellow AIAA.

¶Professor, Department of Mechanical Engineering. Senior Member AIAA.

$$v_{tt} + \delta_e(v_t - \Omega w) - 2\Omega w_t - \Omega^2 v - \delta_{if} v_{xxt} - \frac{\partial}{\partial x}\left\{\frac{c_2^2 + c_1^2[1 + \delta_{im}(\mathrm{d}/\mathrm{d}t)]\left[\sqrt{(1+u_x)^2 + v_x^2 + w_x^2} - 1\right]}{\sqrt{(1+u_x)^2 + v_x^2 + w_x^2}} v_x\right\} = \bar{q}_y \tag{2}$$

$$w_{tt} + \delta_e(w_t + \Omega v) + 2\Omega v_t - \Omega^2 w - \delta_{if} w_{xxt} - \frac{\partial}{\partial x}\left\{\frac{c_2^2 + c_1^2[1 + \delta_{im}(\mathrm{d}/\mathrm{d}t)]\left[\sqrt{(1+u_x)^2 + v_x^2 + w_x^2} - 1\right]}{\sqrt{(1+u_x)^2 + v_x^2 + w_x^2}} w_x\right\} = \bar{q}_z \tag{3}$$

where $(\cdot)_x$, $(\cdot)_t$ imply differentiation with respect to x and t, respectively; c_1 and c_2 are the longitudinal and transverse speeds of sound defined by $c_1^2 = E/\rho$ and $c_2^2 = N_0/(\rho A)$; and the external forces are represented by $\bar{q}_i = q_i/(\rho A)$, $i = x, y, z$. Other symbols in the equations of motion are E, the Young's modulus; A, the cross-sectional area; ρ, the density of the tether; δ_e, the external damping; δ_{im}, the internal damping coefficient of the tether; and δ_{if}, the other internal damping attributable the journal bearing friction. Note that the nonspinning version of Eqs. (1–3) is identical to those given by O'Reilly and Holmes.[16] Assuming a Kelvin–Voigt type of string material with a constitutive equation of the form $\sigma = E(\varepsilon + \delta_{im}\dot{\varepsilon})$, the tension in the tether at a point nominally at x is given by

$$N(x,t) = N_0 + EA\left\{\left[(1+u_x)^2 + v_x^2 + w_x^2\right]^{\frac{1}{2}} - 1 + \delta_{im}\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+u_x)^2 + v_x^2 + w_x^2\right]^{\frac{1}{2}}\right\} \tag{4}$$

where N_0 is the tension at rest. The other internal damping terms attributable to the journal bearing friction, and so forth in x, y, and z directions are assumed to be of the form, $\delta_{if}\{u_{xxt}, v_{xxt}, w_{xxt}\}$, respectively. For a tether with fixed ends as shown in Fig. 1, the boundary conditions are

$$u = v = w = 0 \quad \text{at} \quad x = 0 \text{ and } l \tag{5}$$

To simplify the equations of motion, it is assumed that the interaction between the transverse and longitudinal modes are negligible.[5] Consequently, the longitudinal inertia u_{tt} and its damping can be discarded and, also, c_2^2 is small compared to c_1^2. Assuming that the external forces in the x direction are zero, and expanding by Taylor series, gives

$$v_{tt} + \delta_e(v_t - \Omega w) - \delta_{if} v_{xxt} - 2\Omega w_t - \Omega^2 v - c_2^2 v_{xx} - \frac{c_1^2}{2l}\left(1 + \delta_{im}\frac{\mathrm{d}}{\mathrm{d}t}\right) v_{xx} \int_0^l \left(v_x^2 + w_x^2\right)\mathrm{d}x = \bar{q}_y \tag{6}$$

$$w_{tt} + \delta_e(w_t + \Omega v) - \delta_{if} w_{xxt} + 2\Omega v_t - \Omega^2 w - c_2^2 w_{xx} - \frac{c_1^2}{2l}\left(1 + \delta_{im}\frac{\mathrm{d}}{\mathrm{d}t}\right) w_{xx} \int_0^l \left(v_x^2 + w_x^2\right)\mathrm{d}x = \bar{q}_z \tag{7}$$

For the simply supported ends, the solutions to Eqs. (6) and (7) are taken in the form

$$v = \sum_{n=1}^{\infty} V_n(t)\sin\frac{n\pi x}{l} \quad \text{and} \quad w = \sum_{n=1}^{\infty} W_n(t)\sin\frac{n\pi x}{l} \tag{8}$$

in which n is an integer and V_n, W_n are unknown modal amplitudes. Substituting Eq. (8) into Eqs. (6) and (7), employing the Galerkin method, and considering only separately spatial modes, we get the simplified coupled equations for the tether vibration as

$$\ddot{V}_n + \delta_e(\dot{V}_n - \Omega W_n) + \delta_{if}\omega_n^2 \dot{V}_n - 2\Omega \dot{W}_n + \left(\omega_n^2 - \Omega^2\right)V_n + \alpha_n\left(1 + \delta_{im}\frac{\mathrm{d}}{\mathrm{d}t}\right)\left(V_n^2 + W_n^2\right)V_n = \bar{q}_{yn}(t) \tag{9}$$

$$\ddot{W}_n + \delta_e(\dot{W}_n + \Omega V_n) + \delta_{if}\omega_n^2 \dot{W}_n + 2\Omega \dot{V}_n + \left(\omega_n^2 - \Omega^2\right)W_n + \alpha_n\left(1 + \delta_{im}\frac{\mathrm{d}}{\mathrm{d}t}\right)\left(V_n^2 + W_n^2\right)W_n = \bar{q}_{zn}(t) \tag{10}$$

where

$$\omega_n = \frac{n\pi c_2}{l}; \qquad \alpha_n = \frac{n^2\pi^4 c_1^2}{4l^2}$$
$$\bar{q}_{yn} = \frac{2}{l^2}\int_0^l \bar{q}_y(t)\sin\frac{n\pi x}{l}\,\mathrm{d}x; \qquad \bar{q}_{zn} = \frac{2}{l^2}\int_0^l \bar{q}_z(t)\sin\frac{n\pi x}{l}\mathrm{d}x \tag{11}$$

III. Linear Analysis

From the linear solutions, the resonance and linear stability conditions of a spinning tether are derived for both undamped and damped tethers, for $q_{yn} = Q_y\cos(\Omega_y t)$ and $q_{zn} = Q_z\cos(\Omega_z t)$. Neglecting all nonlinear terms and all forms of internal tether damping in Eqs. (9) and (10), and discarding the subscript n for convenience, the governing differential equations for harmonic forcing reduce to

$$\ddot{V} + \delta_e(\dot{V} - \Omega W) + \delta_{if}\omega^2\dot{V} - 2\Omega\dot{W} + (\omega^2 - \Omega^2)V = Q_y\cos(\Omega_y t) \tag{12}$$

$$\ddot{W} + \delta_e(\dot{W} + \Omega V) + \delta_{if}\omega^2\dot{W} + 2\Omega\dot{V} + (\omega^2 - \Omega^2)W = Q_z\cos(\Omega_z t) \tag{13}$$

A. Free-Vibration Analysis

1. Stability Condition

For this case, $Q_y = Q_z = 0$. Putting Eqs. (12) and (13) into their equivalent standard form produces

$$\begin{aligned} \dot{V} &= X \\ \dot{X} &= 2\Omega Y - (\omega^2 - \Omega^2)V - \left(\delta_e + \omega^2\delta_{if}\right)X + \delta_e\Omega W \\ \dot{W} &= Y \\ \dot{Y} &= -2\Omega X - (\omega^2 - \Omega^2)W - (\delta_e + \omega^2\delta_{if})Y + \delta_e\Omega V \end{aligned} \tag{14}$$

The parameter matrix A is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(\omega^2 - \Omega^2) & -\left(\delta_e + \omega^2\delta_{if}\right) & \delta_e\Omega & 2\Omega \\ 0 & 0 & 0 & 1 \\ -\delta_e\Omega & -2\Omega & -(\omega^2 - \Omega^2) & -\left(\delta_e + \omega^2\delta_{if}\right) \end{bmatrix} \tag{15}$$

The characteristic roots of Eq. (15) are

$$\lambda_{1,2} = \rho_1 \pm \omega_1 i, \qquad \lambda_{3,4} = \rho_2 \pm \omega_2 i \tag{16}$$

where

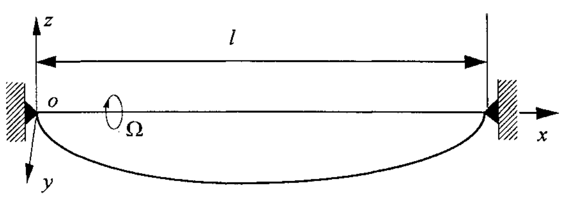

Fig. 1 Mechanical model of the spinning tether.

$$\rho_{1,2}=\frac{-\left(\delta_e+\omega^2\delta_{if}\right)}{2}\pm\frac{1}{2\sqrt{2}}\sqrt{\left\{\left[\left(\delta_e+\omega^2\delta_{if}\right)^2-4\omega^2\right]^2+\left[4\Omega\omega^2\delta_{if}\right]^2\right\}^{\frac{1}{2}}+\left(\delta_e+\omega^2\delta_{if}\right)^2-4\omega^2} \tag{17}$$

$$\omega_{1,2}=\left|\Omega\mp\frac{1}{2\sqrt{2}}\sqrt{\left\{\left[\left(\delta_e+\omega^2\delta_{if}\right)^2-4\omega^2\right]^2+\left[4\Omega\omega^2\delta_{if}\right]^2\right\}^{\frac{1}{2}}-\left(\delta_e+\omega^2\delta_{if}\right)^2-4\omega^2}\right| \tag{18}$$

For stable motion, the real part of the characteristic roots must be zero or negative, that is, $\rho_{1,2}\le 0$. Therefore, the stability condition is

$$(\Omega-\omega)\omega\delta_{if}\le\delta_e \tag{19}$$

Substituting Eq. (11) into Eq. (19) and taking the different modes n into account, we have the stability conditions as

$$\begin{aligned}
&0<\Omega\le\frac{\pi c_2}{l}+\frac{l\delta_e}{\pi c_2\delta_{if}}, && n=1\\
&\frac{(n-1)\pi c_2}{l}+\frac{l\delta_e}{(n-1)\pi c_2\delta_{if}}\\
&\le\Omega\le\frac{n\pi c_2}{l}+\frac{l\delta_e}{n\pi c_2\delta_{if}}, && n=2,3,\ldots
\end{aligned} \tag{20}$$

which are identical to that of Shaw and Shaw.[20] The special case of $(\Omega-\omega)\omega\delta_{if}=\delta_e$ produces

$$\begin{aligned}
\rho_1&=0; & \rho_2&=-\left(\delta_e+\omega^2\delta_{if}\right)\\
\omega_1&=|\Omega-\omega|; & \omega_2&=\Omega+\omega
\end{aligned} \tag{21}$$

Equation (21) implies that a steady periodic solution of the equation of motion (14) can be found, and this will be given later. For the general case of $(\Omega-\omega)\omega\delta_{if}<\delta_e$, the solution of Eq. (14) converges to the point (0, 0, 0, 0) as $t\to\infty$. This implies that the motion of the spinning tether is stable and eventually becomes stationary. On the other hand, if $(\Omega-\omega)\omega\delta_{if}>\delta_e$, the solution of Eq. (14) is divergent and thus unstable. It is of interest to point out two special damped motions:

Case 1. $\delta_e=0$. It is clear from Eq. (19) that the spinning motion is stable if and only if $\Omega\le\omega$. That is, the stability condition of the spinning tether becomes independent of the internal damping. Critical stability is attained when $\Omega=\omega$.

Case 2. $\delta_{if}=0$. For this situation, the solution of Eq. (14) is always stable.

2. *Solution for the Damped Tether*

The homogeneous solution for the set of Eqs. (12) and (13) can be written as

$$\begin{aligned}
V&=e^{\rho_2 t}(A_{d1}\cos\omega_2 t+B_{d1}\sin\omega_2 t)\\
&\quad+e^{\rho_1 t}(A_{d2}\cos\omega_1 t+B_{d2}\sin\omega_1 t)
\end{aligned} \tag{22}$$

$$\begin{aligned}
W&=e^{\rho_2 t}(A_{d1}\sin\omega_2 t-B_{d1}\cos\omega_2 t)\\
&\quad+e^{\rho_1 t}(A_{d2}\sin\omega_1 t-B_{d2}\cos\omega_1 t)
\end{aligned} \tag{23}$$

where the coefficients are defined by

$$\left.\begin{aligned}
A_{dk}&=(-1)^k\frac{(\omega_2-\omega_1)(-Y_0+\omega_k V_0+\rho_1 W_0)+(\rho_1-\rho_2)(X_0-\rho_k V_0+\omega_1 W_0)}{(\omega_2-\omega_1)^2+(\rho_1-\rho_2)^2}\\
B_{dk}&=(-1)^{k+1}\frac{(\omega_2-\omega_1)(X_0+\omega_k W_0-\rho_1 V_0)+(\rho_1-\rho_2)(Y_0-\rho_k W_0-\omega_1 V_0)}{(\omega_2-\omega_1)^2+(\rho_1-\rho_2)^2}
\end{aligned}\right\}\quad k=1,2 \tag{24}$$

It would be of interest to obtain the stable periodic solution of Eq. (14), that is, at $\Omega=\omega+\delta_e/(\omega\delta_{if})$. For this case, its steady-state solution for $\Omega>\omega$ is given by

$$V=V_0\cos(\Omega-\omega)t-W_0\sin(\Omega-\omega)t \tag{25}$$

$$W=W_0\cos(\Omega-\omega)t+V_0\sin(\Omega-\omega)t \tag{26}$$

where V_0 and W_0 are the initial displacements of the steady-state motion and the expressions for their velocities are given by

$$X_0=(\Omega-\omega)W_0,\qquad Y_0=-(\Omega-\omega)V_0 \tag{27}$$

Likewise, one obtains similar results for $\Omega<\omega$.

3. *Solution for the Undamped Tether*

Substituting the conditions for undamped vibration $\rho_1=\rho_2=0$ and $\omega_{1,2}=|\omega\pm\Omega|$ at $\delta_e=\delta_{if}=0$ in Eqs. (12) and (13) for the undamped vibration gives the solutions as

$$\begin{aligned}
V&=A_1\cos(\omega-\Omega)t+A_2\sin(\omega-\Omega)t\\
&\quad+A_3\cos(\omega+\Omega)t+A_4\sin(\omega+\Omega)t
\end{aligned} \tag{28}$$

$$\begin{aligned}
W&=-A_2\cos(\omega-\Omega)t+A_1\sin(\omega-\Omega)t\\
&\quad-A_4\cos(\omega+\Omega)t+A_3\sin(\omega+\Omega)t
\end{aligned} \tag{29}$$

where

$$\left.\begin{aligned}
A_1&=\frac{(\Omega+\omega)V_0-Y_0}{2\omega}\\
A_2&=-\frac{(\Omega+\omega)W_0+X_0}{2\omega}\\
A_3&=-\frac{(\Omega-\omega)V_0-Y_0}{2\omega}\\
A_4&=\frac{(\Omega-\omega)W_0+X_0}{2\omega}
\end{aligned}\right\}\quad\text{for}\quad\omega<\Omega \tag{30}$$

$$\left.\begin{aligned}
A_1&=\frac{(\omega+\Omega)V_0-Y_0}{2\Omega}\\
A_2&=-\frac{(\omega+\Omega)W_0+X_0}{2\Omega}\\
A_3&=-\frac{(\omega-\Omega)V_0-Y_0}{2\Omega}\\
A_4&=\frac{(\omega-\Omega)W_0+X_0}{2\Omega}
\end{aligned}\right\}\quad\text{for}\quad\omega>\Omega \tag{31}$$

with V_0, X_0, W_0, and Y_0 as initial conditions. Thus, in the phase space, we have

$$[(\omega-\Omega)W+X]^2+[(\omega-\Omega)V-Y]^2 = [(\omega-\Omega)W_0+X_0]^2+[(\omega-\Omega)V_0-Y_0]^2 \tag{32}$$

$$[(\omega+\Omega)W+X]^2+[(\omega+\Omega)V-Y]^2 = [(\omega+\Omega)W_0+X_0]^2+[(\omega+\Omega)V_0-Y_0]^2 \tag{33}$$

Observe that for the particular situation of $\Omega=\omega$ and if $X_0 = Y_0 = 0$, the tether becomes stationary during the entire rotation. That is, its static state is at $V = V_0$, $W = W_0$. Similarly, from the solution, the resonance condition for the undamped spinning tether can be written as

$$\Omega=\frac{m_1+n_1}{m_1-n_1}\omega, \qquad \text{for} \quad \omega<\Omega \tag{34}$$

$$\Omega=\frac{m_1-n_1}{m_1+n_1}\omega, \qquad \text{for} \quad \omega>\Omega \tag{35}$$

Note that m_1 and n_1 are irreducible integers. The resonance conditions derived here are commonly known as the internal resonance conditions.

B. Forced-Vibration Analysis

For the harmonic forcing of a damped tether, particular solutions of Eqs. (12) and (13) are given by

$$V_p=\bar{A}_{d1}\cos\Omega_y t+\bar{A}_{d2}\sin\Omega_y t+\bar{A}_{d3}\cos\Omega_z t+\bar{A}_{d4}\sin\Omega_z t \tag{36}$$

$$W_p=\bar{B}_{d1}\cos\Omega_y t+\bar{B}_{d2}\sin\Omega_y t+\bar{B}_{d3}\cos\Omega_z t+\bar{B}_{d4}\sin\Omega_z t \tag{37}$$

where the various coefficients are defined by

$$\bar{A}_{d1}=\frac{a_4Q_y}{a_1a_4-a_2a_3}; \qquad \bar{A}_{d2}=-\frac{a_3Q_y}{a_1a_4-a_2a_3}$$
$$\bar{A}_{d3}=\frac{(\alpha_2\eta_2-\beta_2\gamma_2)\bar{B}_3+(\gamma_2\alpha_2-\beta_2\eta_2)\bar{B}_4}{\alpha_2^2+\beta_2^2} \tag{38}$$
$$\bar{A}_{d4}=\frac{(\alpha_2\eta_2+\beta_2\gamma_2)\bar{B}_4+(\beta_2\eta_2+\gamma_2\alpha_2)\bar{B}_3}{\alpha_2^2+\beta_2^2}$$

with

$$a_1=\alpha_1+\frac{\alpha_1(\eta_1^2-\gamma_1^2)+2\gamma_1\eta_1\beta_1}{\alpha_1^2+\beta_1^2}$$
$$a_2=\beta_1-\frac{\beta_1(\eta_1^2-\gamma_1^2)-2\gamma_1\eta_1\alpha_1}{\alpha_1^2+\beta_1^2}$$
$$a_3=-\beta_1+\frac{\beta_1(\eta_1^2-\gamma_1^2)-2\gamma_1\eta_1\alpha_1}{\alpha_1^2+\beta_1^2} \tag{40}$$
$$a_4=\alpha_1+\frac{\alpha_1(\eta_1^2-\gamma_1^2)-2\gamma_1\eta_1\beta_1}{\alpha_1^2+\beta_1^2}$$

$$b_1=\alpha_2+\frac{\alpha_2(\eta_2^2+\gamma_2^2)}{\alpha_2^2+\beta_2^2}$$
$$b_2=\beta_2-\frac{\beta_2(\eta_2^2-\gamma_2^2)-2\gamma_2\eta_2\alpha_2}{\alpha_2^2+\beta_2^2}$$
$$b_3=-\beta_2+\frac{\beta_2(\eta_2^2+\gamma_2^2)}{\alpha_2^2+\beta_2^2} \tag{41}$$
$$b_4=\alpha_2-\frac{\alpha_2(\eta_2^2-\gamma_2^2)-2\gamma_2\eta_2\beta_2}{\alpha_2^2+\beta_2^2}$$

and

$$\alpha_i=-\Omega_j^2+(\omega^2-\Omega^2); \qquad \beta_i=(\delta_e+\delta_{if}\omega^2)\Omega_j$$
$$\gamma_i=2\Omega\Omega_j; \qquad \eta_i=\delta_e\Omega_j \tag{42}$$

Note that in Eq. (42), when $i=1$, $j=y$ and likewise, when $i=2$, $j=z$. For the damped motion of a spinning tether, the total solutions can therefore be written as

$$V=e^{\rho_2 t}(A_{d1}\cos\omega_2 t+B_{d1}\sin\omega_2 t) + e^{\rho_1 t}(A_{d2}\cos\omega_1 t+B_{d2}\sin\omega_1 t)+V_p \tag{43}$$

$$W=e^{\rho_2 t}(A_{d1}\sin\omega_2 t-B_{d1}\cos\omega_2 t) + e^{\rho_1 t}(A_{d2}\sin\omega_1 t-B_{d2}\cos\omega_1 t)+W_p \tag{44}$$

where the coefficients of Eqs. (43) and (44) are given by

$$\left.\begin{aligned} A_{dk}&=(-1)^k\frac{(\omega_2-\omega_1)(-\hat{Y}_0+\omega_k\hat{V}_0+\rho_1\hat{W}_0)+(\rho_1-\rho_2)(\hat{X}_0-\rho_k\hat{V}_0+\omega_1\hat{W}_0)}{(\omega_2-\omega_1)^2+(\rho_1-\rho_2)^2}\\ B_{dk}&=(-1)^{k+1}\frac{(\omega_2-\omega_1)(\hat{X}_0+\omega_k\hat{W}_0-\rho_1\hat{V}_0)+(\rho_1-\rho_2)(\hat{Y}_0-\rho_k\hat{W}_0-\omega_1\hat{V}_0)}{(\omega_2-\omega_1)^2+(\rho_1-\rho_2)^2}\end{aligned}\right\} \quad k=1,2 \tag{45}$$

$$\bar{B}_{d1}=-\frac{(\alpha_1\eta_1+\beta_1\gamma_1)\bar{A}_1+(\gamma_1\alpha_1-\beta_1\eta_1)\bar{A}_2}{\alpha_1^2+\beta_1^2}$$
$$\bar{B}_{d2}=-\frac{(\alpha_1\eta_1+\beta_1\gamma_1)\bar{A}_2+(\beta_1\eta_1-\gamma_1\alpha_1)\bar{A}_1}{\alpha_1^2+\beta_1^2} \tag{39}$$
$$\bar{B}_{d3}=\frac{b_4Q_z}{b_1b_4-b_2b_3}; \qquad \bar{B}_{d4}=-\frac{b_3Q_z}{b_1b_4-b_2b_3}$$

and

$$\hat{V}_0=V_0-\bar{A}_1-\bar{A}_3; \qquad \hat{X}_0=X_0-\Omega_y\bar{A}_2-\Omega_z\bar{A}_4$$
$$\hat{W}_0=W_0-\bar{B}_1-\bar{B}_3; \qquad \hat{Y}_0=Y_0-\Omega_y\bar{B}_2-\Omega_z\bar{B}_4 \tag{46}$$

Equations (43) and (44) constitute the most general solution for the motion of a stretched spinning tether. For the case of

$(\Omega-\omega)\omega\delta_{if} < \delta_e$ and if the transient solution is not considered, there will be only the external excitation resonance. For the case of $(\Omega-\omega)\omega\delta_{if} > \delta_e$, there are no stable solutions. For the particular case of $(\Omega-\omega)\omega\delta_{if} = \delta_e$, its steady solutions can be given by

$$V = \hat{A}_1\cos(\omega_1 t) + \hat{A}_2\sin(\omega_1 t) + \bar{A}_1\cos(\Omega_y t) + \bar{A}_2\sin(\Omega_y t) + \bar{A}_3\cos(\Omega_z t) + \bar{A}_4\sin(\Omega_z t) \tag{47}$$

$$W = -\hat{A}_2\cos(\omega_1 t) + \hat{A}_1\sin(\omega_1 t) + \bar{B}_1\cos(\Omega_y t) + \bar{B}_2\sin(\Omega_y t) + \bar{B}_3\cos(\Omega_z t) + \bar{B}_4\sin(\Omega_z t) \tag{48}$$

where $\omega_1 = |\Omega-\omega|$ and the coefficients $\hat{A}_1$, $\hat{A}_2$ are defined by

$$\hat{A}_1 = V_0 - \bar{A}_1 - \bar{A}_3, \qquad \hat{A}_2 = W_0 - \bar{B}_1 - \bar{B}_3 \tag{49}$$

The external resonance conditions are

$$\{\Omega_y, \Omega_z\} = \Omega - \omega \qquad \text{for} \quad \omega < \Omega \tag{50}$$

$$\{\Omega_y, \Omega_z\} = \omega - \Omega \qquad \text{for} \quad \omega > \Omega \tag{51}$$

and its subharmonic resonance conditions are

$$\{\Omega_y, \Omega_z\} = (n_2/m_2)(\Omega-\omega) \qquad \text{for} \quad \omega < \Omega \tag{52}$$

$$\{\Omega_x, \Omega_y\} = (n_2/m_2)(\omega-\Omega) \qquad \text{for} \quad \omega > \Omega \tag{53}$$

For an undamped tether, using $\rho_1 = \rho_2 = 0$ and $\omega_{1,2} = |\omega \pm \Omega|$ at $\delta_e = \delta_{if} = 0$, particular solutions are obtained as

$$V_p = \bar{A}_1\cos\Omega_y t + \bar{A}_2\sin\Omega_z t \tag{54}$$

$$W_p = \bar{A}_3\sin\Omega_y t + \bar{A}_4\cos\Omega_z t \tag{55}$$

where

$$\bar{A}_1 = \frac{\alpha_1}{\alpha_1^2-\alpha_2^2}Q_y; \qquad \bar{A}_2 = \frac{\alpha_4}{\alpha_3^2-\alpha_4^2}Q_z$$

$$\bar{A}_3 = \frac{\alpha_2}{\alpha_1^2-\alpha_2^2}Q_y; \qquad \bar{A}_4 = \frac{\alpha_3}{\alpha_3^2-\alpha_4^2}Q_z$$

and

$$\begin{aligned} \alpha_1 &= -\Omega_y^2 + (\omega^2-\Omega^2); \qquad & \alpha_2 &= 2\Omega\Omega_y \\ \alpha_3 &= -\Omega_z^2 + (\omega^2-\Omega^2); \qquad & \alpha_4 &= 2\Omega\Omega_z \end{aligned} \tag{56}$$

The resonance condition is obtained by setting $\alpha_1 = \pm\alpha_2$ and $\alpha_3 = \pm\alpha_4$ in Eq. (56), and this yields

$$\{\Omega_y, \Omega_z\} = \{\Omega-\omega \quad \text{or} \quad \omega+\Omega\} \qquad \text{for} \quad \omega < \Omega \tag{57}$$

$$\{\Omega_y, \Omega_z\} = \{\omega-\Omega \quad \text{or} \quad \omega+\Omega\} \qquad \text{for} \quad \omega > \Omega \tag{58}$$

Unlike in the free-vibration analysis, here they are referred to as the external resonance conditions. The complete solution to the undamped case [Eqs. (12) and (13)] includes both the homogeneous and particular components. The coefficients of the former in Eqs. (28) and (29) are

$$\left.\begin{aligned} A_1 &= \frac{(\omega+\Omega)(V_0-\bar{A}_1) - Y_0 - \bar{A}_3\Omega_y}{2\omega} \\ A_2 &= -\frac{X_0 + \Omega_z\bar{A}_2 + (\omega+\Omega)(W_0-\bar{A}_4)}{2\omega} \\ A_3 &= -\frac{(\Omega-\omega)(V_0-\bar{A}_1) - Y_0 - \bar{A}_3\Omega_y}{2\omega} \\ A_4 &= \frac{X_0 + \Omega_z\bar{A}_2 + (\Omega-\omega)(W_0-\bar{A}_4)}{2\omega} \end{aligned}\right\} \quad \text{for } \omega < \Omega \tag{59}$$

$$\left.\begin{aligned} A_1 &= \frac{(\omega+\Omega)(V_0-\bar{A}_1) - Y_0 - \bar{A}_3\Omega_y}{2\Omega} \\ A_2 &= -\frac{X_0 + \Omega_z\bar{A}_2 + (\omega+\Omega)(W_0-\bar{A}_4)}{2\Omega} \\ A_3 &= -\frac{(\omega-\Omega)(V_0-\bar{A}_1) + Y_0 + \bar{A}_3\Omega_y}{2\Omega} \\ A_4 &= \frac{X_0 + \Omega_z\bar{A}_2 + (\omega-\Omega)(W_0-\bar{A}_4)}{2\Omega} \end{aligned}\right\} \quad \text{for } \omega > \Omega \tag{60}$$

In addition to the main resonance, there is a subharmonic resonance attributable to external excitation, the conditions for which are given by

$$\{\Omega_y, \Omega_z\} = \{(n_2/m_2)(\Omega-\omega) \quad \text{or} \quad (n_2/m_2)(\omega+\Omega)\} \qquad \text{for} \quad \omega < \Omega \tag{61}$$

$$\{\Omega_y, \Omega_z\} = \{(n_2/m_2)(\omega-\Omega) \quad \text{or} \quad (n_2/m_2)(\omega+\Omega)\} \qquad \text{for} \quad \omega > \Omega \tag{62}$$

where m_2 and n_2 are irreducible integers. When there are two or more excitations acting on the system, external excitation resonance occurs. The condition for this external excitation resonance is derived as

$$m_3\Omega_y = n_3\Omega_z \tag{63}$$

where m_3 and n_3 are irreducible integers except at the main resonance when $m_3 = n_3 = 1$.

IV. Nonlinear Analysis

In this section, resonant conditions for the undamped tether are presented and the solution for free vibration of the nonlinear spinning tether is obtained via an energy analysis. Consider Eqs. (9) and (10) with $q_{yn} = Q_y\cos(\Omega_y t)$ and $q_{zn} = Q_z\cos(\Omega_z t)$ for the undamped case to obtain resonant conditions. As before, discarding the subscript n in Eqs. (9) and (10), the system Hamiltonian is given by

$$H = \tfrac{1}{2}(X^2+Y^2) + \tfrac{1}{2}(\omega^2-\Omega^2)(V^2+W^2) + \tfrac{1}{4}\alpha(V^2+W^2)^2 - VQ_y\cos\Omega_y t - WQ_z\cos\Omega_z t \tag{64}$$

Note that the Hamiltonian can be separated into the non-time-dependent part H_0 and the time-dependent part H_1. That is, $H = H_0 + H_1$, where

$$H_0 = \tfrac{1}{2}(X^2+Y^2) + \tfrac{1}{2}(\omega^2-\Omega^2)(V^2+W^2) + \tfrac{1}{4}\alpha(V^2+W^2)^2 \tag{65}$$

$$H_1 = -VQ_y\cos\Omega_y t - WQ_z\cos\Omega_z t \tag{66}$$

Setting the non-time-dependent Hamiltonian H_0 to take on a particular initial energy E_0, solutions for the free vibration of the undamped spinning tether have the form[21,22]

$$V = \frac{\pi h}{2kK(k)}\sum_{m=0}^{\infty}\operatorname{sech}\left[\left(m+\frac{1}{2}\right)\pi\frac{K(k')}{K(k)}\right] \times \{\cos[(2m+1)\omega' + \Omega]t + \cos[(2m+1)\omega' - \Omega]t\} \tag{67}$$

$$W = \frac{\pi h}{2kK(k)}\sum_{m=0}^{\infty}\operatorname{sech}\left[\left(m+\frac{1}{2}\right)\pi\frac{K(k')}{K(k)}\right] \times \{\cos[(2m+1)\omega' - \Omega]t - \cos[(2m+1)\omega' + \Omega]t\} \tag{68}$$

$$X = \pm\sqrt{\frac{\alpha}{2}}\frac{h^2\pi^2}{8k[K(k)]^2}\sum_{m=0}^{\infty}\left(\begin{array}{l}\frac{1}{2}\operatorname{csch}\left[\left(m+\frac{1}{2}\right)\pi\frac{K(k')}{K(k)}\right]\{\cos[(2m+1)\omega'+\Omega]t+\cos[(2m+1)\omega'-\Omega]t\} \\ +\sum_{j=0}^{\infty}\operatorname{sech}\left[\left(j+\frac{1}{2}\right)\pi\frac{K(k')}{K(k)}\right]\operatorname{csch}\left[\left(m+\frac{1}{2}\right)\pi\frac{K(k')}{K(k)}\right]\left\{\begin{array}{l}\sin[2(m+j+1)\omega'+\Omega]t \\ +\sin[2(m+j+1)\omega'-\Omega]t \\ +\sin[2(m-j)\omega'+\Omega]t \\ +\sin[2(m-j)\omega'-\Omega]t\end{array}\right\}\end{array}\right)$$

$$+\frac{\pi h\Omega}{2kK(k)}\sum_{m=0}^{\infty}\operatorname{sech}\left[\left(m+\frac{1}{2}\right)\pi\frac{K(k')}{K(k)}\right]\{\cos[(2m+1)\omega'-\Omega]t-\cos[(2m+1)\omega'+\Omega]t\} \tag{69}$$

$$Y = \pm\sqrt{\frac{\alpha}{2}}\frac{h^2\pi^2}{8k[K(k)]^2}\sum_{m=0}^{\infty}\left(\begin{array}{l}\frac{1}{2}\operatorname{csch}\left[\left(m+\frac{1}{2}\right)\pi\frac{K(k')}{K(k)}\right]\{\sin[(2m+1)\omega'+\Omega]t+\sin[(2m+1)\omega'-\Omega]t\} \\ +\sum_{j=0}^{\infty}\operatorname{sech}\left[\left(j+\frac{1}{2}\right)\pi\frac{K(k')}{K(k)}\right]\operatorname{csch}\left[\left(m+\frac{1}{2}\right)\pi\frac{K(k')}{K(k)}\right]\left\{\begin{array}{l}\cos[2(m+j+1)\omega'-\Omega]t \\ +\sin[2(m+j+1)\omega'+\Omega]t \\ +\cos[2(m-j)\omega'-\Omega]t \\ -\cos[2(m-j)\omega'+\Omega]t\end{array}\right\}\end{array}\right)$$

$$-\frac{\pi h\Omega}{2kK(k)}\sum_{m=0}^{\infty}\operatorname{sech}\left[\left(m+\frac{1}{2}\right)\pi\frac{K(k')}{K(k)}\right]\{\cos[(2m+1)\omega'+\Omega]t+\cos[(2m+1)\omega'-\Omega]t\} \tag{70}$$

in which $K(k)$ is the complete elliptic integral of the first kind, k is the modulus of the Jacobian-elliptic function, and $k' = \sqrt{(1-k^2)}$. The parameter h of the solution is defined by

$$h^2 = \frac{2k^2\omega^2}{(1-2k^2)\alpha} \tag{71}$$

The nonlinear frequency of the undamped spinning tether ω' is computed from

$$\omega' = \frac{\pi}{2K(k)\sqrt{1-2k^2}}\omega \tag{72}$$

Observe that as $k \to 0$, $K(k) \to \pi/2$ and $\omega' \to \omega$. That is, the linear frequency can be recovered from the nonlinear result in Eq. (72). The particular value of $H_0 = E_0$ can be obtained via

$$E_0 = \frac{k^2(1-k^2)\omega^4}{(1-2k^2)^2\alpha} \tag{73}$$

and the action variable J at an orbit surface is given by

$$J = \frac{\sqrt{2\alpha}h^3}{3\pi k^3}\left[(1-k^2)K(k)+(2k^2-1)E(k)\right] \tag{74}$$

where $E(k)$ denotes the complete elliptic integral of the second kind. The period T is given by

$$T = 2\pi/\omega' \tag{75}$$

Substituting Eqs. (67–70) into Eq. (64), the complete Hamiltonian now becomes

$$H = H_0(J) - Q_y\sum_{m=0}^{\infty}Q_{2m+1}\left\{\begin{array}{l}\cos[(2m+1)\omega'+\Omega+\Omega_y]t+\cos[(2m+1)\omega'-\Omega+\Omega_y]t \\ +\cos[(2m+1)\omega'+\Omega-\Omega_y]t+\cos[(2m+1)\omega'-\Omega-\Omega_y]t\end{array}\right\}$$

$$-Q_z\sum_{m=0}^{\infty}Q_{2m+1}\left\{\begin{array}{l}\sin[(2m+1)\omega'+\Omega+\Omega_z]t+\sin[(2m+1)\omega'-\Omega+\Omega_z]t \\ +\sin[(2m+1)\omega'+\Omega-\Omega_z]t+\sin[(2m+1)\omega'-\Omega-\Omega_z]t\end{array}\right\} \tag{76}$$

with

$$Q_{2m+1} = \frac{\pi h}{2kK\cosh\left[\pi\left(m+\frac{1}{2}\right)(K'/K)\right]} \tag{77}$$

where $K' = K(k')$ and $k' = 1-k^2$. As in the linear case, Eqs. (67) and (68) yield the following internal resonant conditions:

$$\Omega = \frac{m_1+n_1}{m_1-n_1}(2m+1)\omega' \quad \text{for} \quad (2m+1)\omega' < \Omega \tag{78}$$

$$\Omega = \frac{m_1-n_1}{m_1+n_1}(2m+1)\omega' \quad \text{for} \quad (2m+1)\omega' > \Omega \tag{79}$$

From Eq. (76), the subharmonic resonances attributable to external excitation are

$$\{\Omega_y,\Omega_z\} = \{(n_2/m_2)[\Omega-(2m+1)\omega'] \text{ or } (n_2/m_2)[\Omega+(2m+1)\omega']\} \quad \text{for} \quad (2m+1)\omega' < \Omega \tag{80}$$

$$\{\Omega_y,\Omega_z\} = \{(n_2/m_2)[(2m+1)\omega'-\Omega] \text{ or } (n_2/m_2)[(2m+1)\omega'+\Omega]\} \quad \text{for} \quad (2m+1)\omega' > \Omega \tag{81}$$

Setting $k = 0$ and $m = 0$ in Eqs. (78) and (79), the corresponding results for linear internal resonance [Eqs. (34) and (35)] are recovered. Likewise, the linear subharmonic resonance [Eqs. (61) and (62)] also can be recovered from Eqs. (80) and (81).

V. Numerical Simulations

To demonstrate the procedure, numerical simulations of linear and nonlinear systems via a Runge–Kutta integration scheme are carried out.

A. Linear Systems

For linear systems, resonance graphs pertaining to undamped free vibrations, undamped forced vibrations, and damped forced vibrations are presented in Figs. 2, 3, and 4, respectively. Their input parameters are summarized in Table 1. For the undamped motion, its internal resonance is depicted in Fig. 2, and external excitation resonance in Fig. 3. For the damped motion, its external excitation

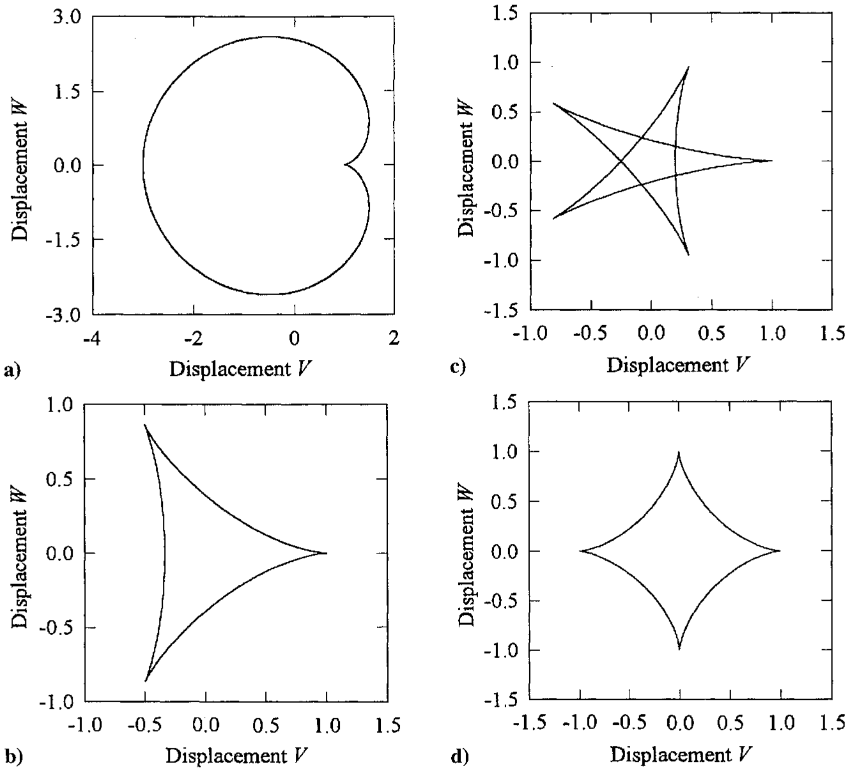

Fig. 2 Resonance motion for undamped free vibrations.

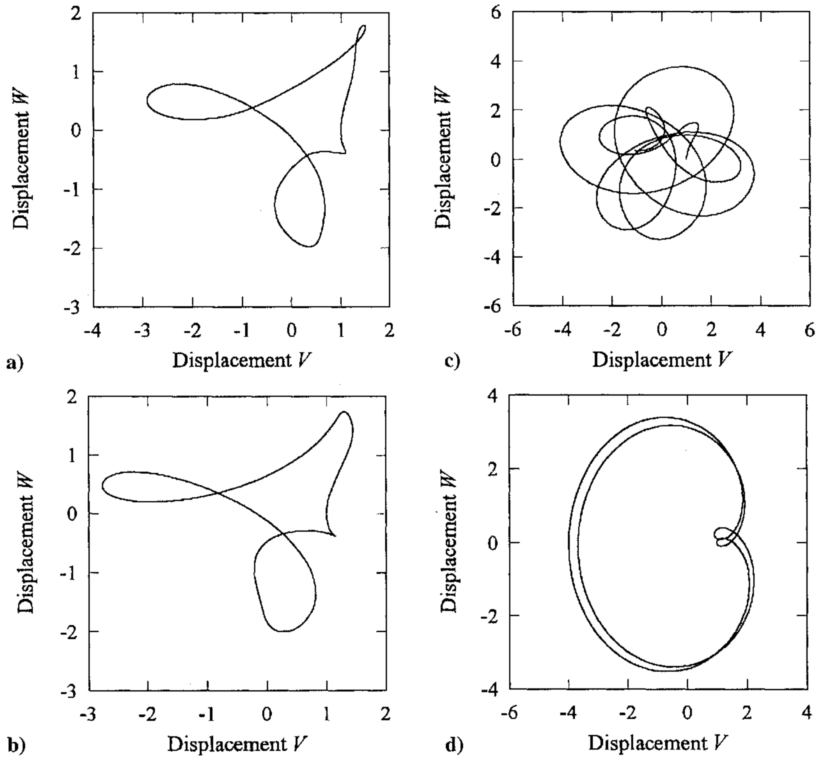

Fig. 3 Resonance motion for undamped forced vibrations.

resonance is given in Fig. 4. Observe that the resonant motion for the undamped vibrations in Figs. 2 and 3 is always stable, which is expected. On the other hand, for the damped forced vibrations, the resonant motion can be unstable, which is clearly evident in Fig. 4c. The other three cases in Figs. 4a, 4b, and 4d all converged to their own resonant limit cycles, and are thus stable.

B. Nonlinear Systems

For nonlinear systems, the free vibration and forced vibrations of an undamped spinning tether are simulated. The input parameters are summarized in Table 2. On the basis of the initial conditions, the nonlinear natural frequency is computed as $\omega' = 2.548389$. The results of the numerical simulation, which are presented in the form of displacement and phase-plane plots, are given in Figs. 5 and 6. In Figs. 5a and 5b (and in Figs. 6a and 6b), the resonant and nonresonant motions for free vibration are depicted, whereas in Figs. 5c and 5d (and in Figs. 6c and 6d) the resonant and nonresonant motions corresponding to forced vibration are shown. From these plots, it is obvious that the motions are always stable, which is expected for the undamped tether.

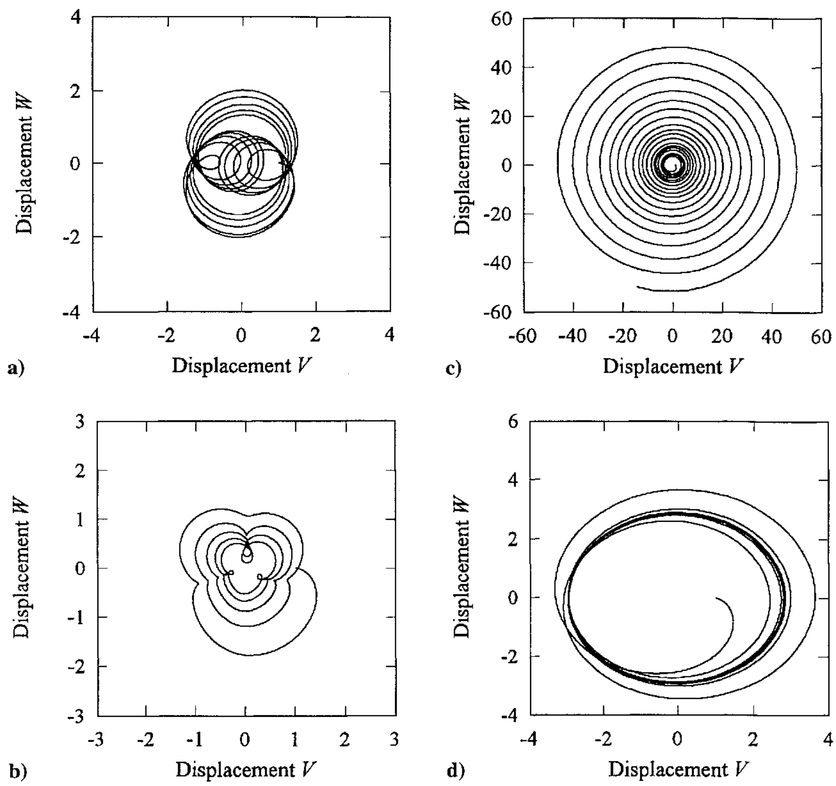

Fig. 4 Resonance motion for damped forced vibrations.

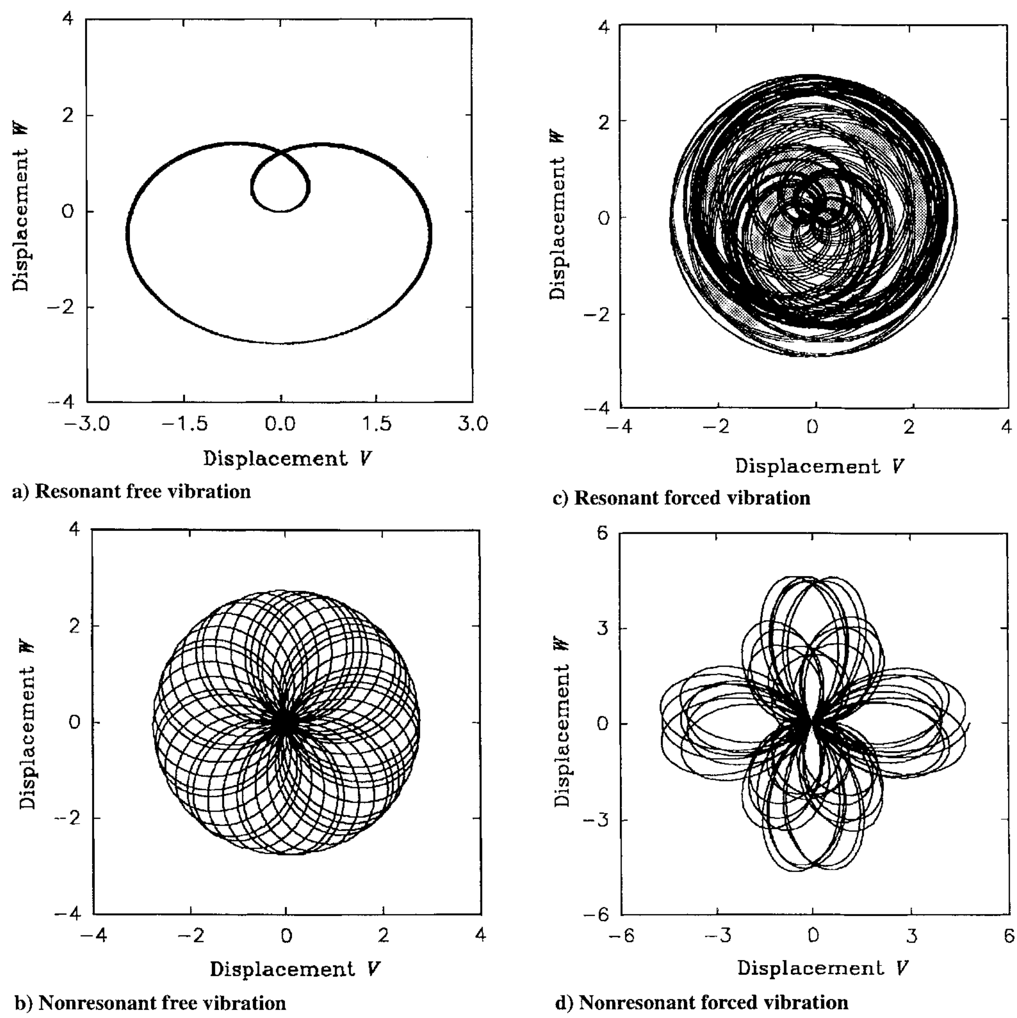

Fig. 5 Displacement plane plots.

Table 1 Input parameters for the linear analysis

Figures	Input parameters
2	$Q_y = Q_z = 0.0$, $V_0 = 1.0$, $X_0 = W_0 = Y_0 = 0.0$, $\omega = 1.0$:
	a) $\Omega = 5.0$, $m_1 = 3$, $n_1 = 2$
	b) $\Omega = \frac{1}{3}$, $m_1 = 2$, $n_1 = 1$
	c) $\Omega = 0.2$, $m_1 = 3$, $n_1 = 2$
	d) $\Omega = 0.5$, $m_1 = 3$, $n_1 = 1$
3	$Q_y = Q_z = 1.0$, $V_0 = Y_0 = 1.0$, $X_0 = W_0 = 0.0$, $\omega = 1.0$:
	a) $\Omega = 0.5$, $\Omega_y = 4.0$, $\Omega_z = 1.0$, $m_1 = 3$, $n_1 = 1$, $m_2 = 1$, $n_2 = 8$, $m_3 = 1$, $n_3 = 4$
	b) $\Omega = 0.5$, $\Omega_y = 3.0$, $\Omega_z = 1.0$, $m_1 = 3$, $n_1 = 1$, $m_2 = 1$, $n_2 = 2$, $m_3 = 1$, $n_3 = 3$
	c) $\Omega = 0.5$, $\Omega_y = 2.7$, $\Omega_z = 1.3$, $m_1 = 3$, $n_1 = 1$, $m_2 = 5$, $n_2 = 27$, $m_3 = 13$, $n_3 = 27$
	d) $\Omega = 3.0$, $\Omega_y = 6.0$, $\Omega_z = 5.0$, $m_1 = 2$, $n_1 = 1$, $m_2 = 2$, $n_2 = 1$, $m_3 = 5$, $n_3 = 6$
4	$Q_y = Q_z = 1.0$, $V_0 = 1.0$, $X_0 = Y_0 = W_0 = 0.0$, $\omega = 1.0$, $\delta_{if} = 0.1$:
	a) $\delta_e = 0.2$, $\Omega = 2.0$, $\Omega_y = 6.0$, $\Omega_z = 3.0$, $m_1 = 3$, $n_1 = 1$, $m_2 = 6$, $n_2 = 1$, $m_3 = 1$, $n_3 = 2$
	b) $\delta_e = 0.2$, $\Omega = 2.0$, $\Omega_y = 5.0$, $\Omega_z = 4.0$, $m_1 = 3$, $n_1 = 1$, $m_2 = 1$, $n_2 = 5$, $m_3 = 4$, $n_3 = 5$
	c) $\delta_e = 0.2$, $\Omega = 5.0$, $\Omega_y = 7.0$, $\Omega_z = 3.0$, $m_1 = 3$, $n_1 = 2$, $m_2 = 4$, $n_2 = 7$, $m_3 = 3$, $n_3 = 7$
	d) $\delta_e = 0.4$, $\Omega = 5.0$, $\Omega_y = 8.0$, $\Omega_z = 12.0$, $m_1 = 3$, $n_1 = 2$, $m_2 = 1$, $n_2 = 2$, $m_3 = 3$, $n_3 = 2$

Table 2 Input parameters for the nonlinear analysis

Figures	Input parameters
5, 6	$Q_z = 0.0$, $V_0 = W_0 = Y_0 = 0.0$, $X_0 = 6.041523$, $\omega = \alpha = 1.0$:
	a) $Q_y = 0.0$, $\Omega = 7.645169$, $m_1 = 2$, $n_1 = 1$, $m = 0$
	b) $Q_y = 0.0$, $\Omega = 2.0$
	c) $Q_y = 6.414878$, $\Omega = 12.74295$, $\Omega_y = 7.645169$, $m_2 = 2$, $n_2 = 3$, $m = 1$
	d) $Q_y = 6.414878$, $\Omega = 2.0$, $\Omega_y = 3.0$

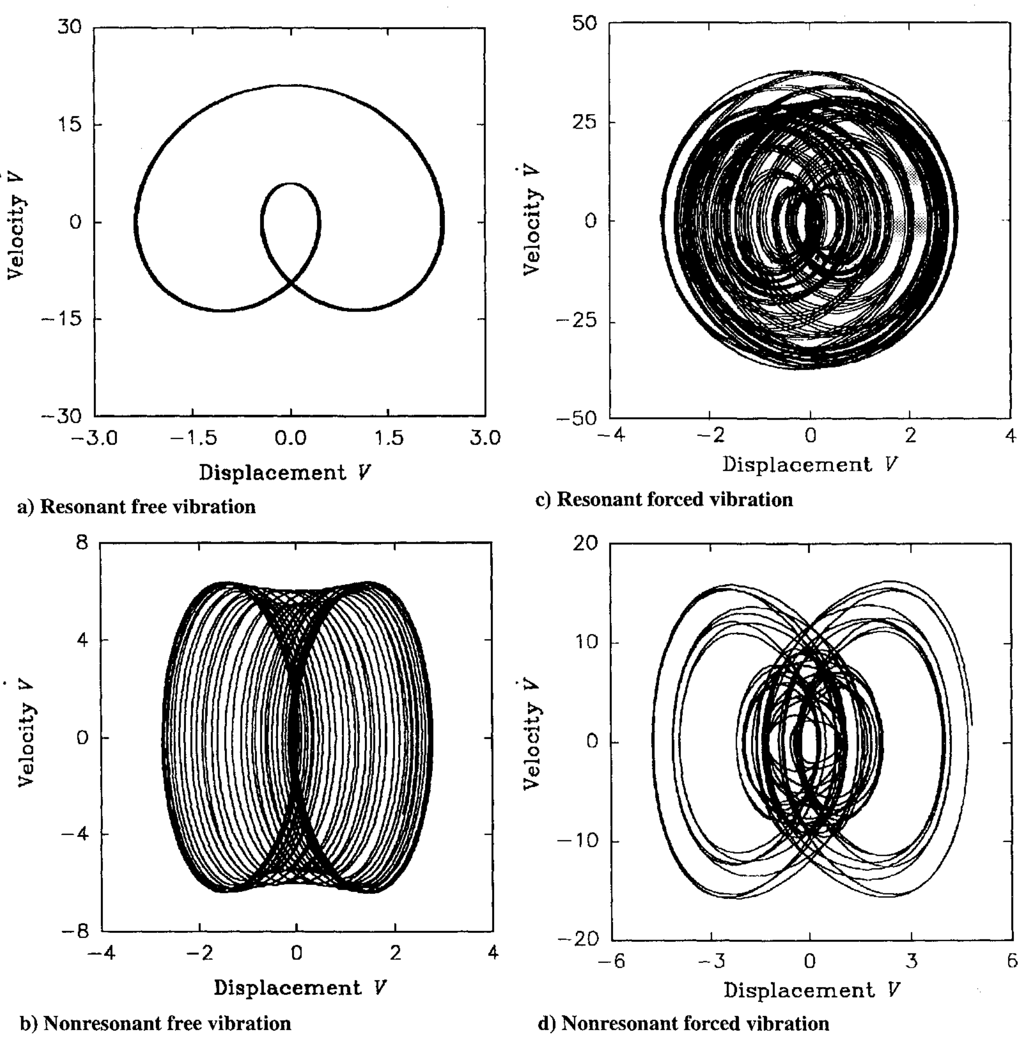

Fig. 6 Phase-plane plots.

VI. Conclusions

In this work, the complete exact solutions for damped and undamped free and forced vibrations for a linear spinning stretched tether are presented. From these results, the resonant conditions for various types of resonance are given. Because the damped motion can be unstable, its stability condition is obtained. For the nonlinear tether, analytical solutions for the undamped free vibration are derived and results for the internal and the subharmonic resonance given. On the other hand, for the nonlinear undamped forced vibration, the results are simulated via numerical integration. Both resonant and nonresonant motions are depicted.

Acknowledgments

Financial support from the Canadian Space Agency (CSA) and Bristol Aerospace Ltd. through Contract 70.08.300.1 are gratefully acknowledged. We would also like to thank F. Vigneron and A. Jablonski of the CSA for their interest in this work.

References

[1]Tyc, G., Han, R. P. S., Vigneron, F. R., Jablonski, A. M., Modi, V. J., and Misra, A. K., "Dynamics and Stability of a Spinning Tethered Spacecraft with Flexible Appendages," AAS/AIAA Astrodynamics Specialist Conf., AAS Paper 93-736 Victoria, BC, Canada, Aug. 1993.

[2]Tyc, G., and Han, R. P. S., "Attitude Dynamics Investigation of the OEDIPUS-A Tethered Rocket Payload," *Journal of Spacecraft and Rockets*, Vol. 32, No. 1, 1995, pp. 133–141.

[3]Carrier, G. F., "On the Non-Linear Vibration Problem of the Elastic String," *Quarterly of Applied Mathematics*, Vol. 3, 1945, pp. 157–165.

[4]Oplinger, D. W., "Frequency Response of a Nonlinear Stretched String," *Journal of the Acoustical Society of America*, Vol. 32, 1960, pp. 1529–1538.

[5]Narashima, R., "Nonlinear Vibration of an Elastic String," *Journal of Sound and Vibration*, Vol. 8, 1968, pp. 134–146.

[6]Anand, G. V., "Large-Amplitude Damped Free Vibration of a Stretched String," *Journal of the Acoustical Society of America*, Vol. 45, 1969, pp. 1089–1096.

[7]Anand, G. V., "Stability of Nonlinear Oscillations of a Stretched String," *Journal of the Acoustical Society of America*, Vol. 46, 1969, pp. 667–677.

[8]Hayashi, C., *Nonlinear Oscillation in Physical Systems*, McGraw–Hill, New York, 1964.

[9]Antman, S. S., "The Equations for Large Vibration of String," *American Mathematical Monthly*, Vol. 87, 1980, pp. 359–370.

[10]Gough, C., "The Nonlinear Free Vibration of a Damped Elastic String," *Journal of the Acoustical Society of America*, Vol. 75, 1984, pp. 1770–1776.

[11]Nayfeh, A. H., and Mook, D. T., *Nonlinear Oscillations*, Wiley, New York, 1979.

[12]Miles, J., "Resonant, Nonplanar Motion of a Stretched String," *Journal of the Acoustical Society of America*, Vol. 75, 1984, pp. 1505–1510.

[13]Tufillaro, N. B., "Nonlinear and Chaotic String Vibrations," *American Journal of Physics*, Vol. 57, 1988, pp. 409–414.

[14]Johnson, J. M., and Bajaj, A. K., "Amplitude Modulated and Chaotic Dynamics in Resonant Motion of Strings," *Journal of Sound and Vibration*, Vol. 128, 1989, pp. 87–107.

[15]Gottlieb, H. P. W., "Nonlinear Vibration of a Constant-Tension String," *Journal of Sound and Vibration*, Vol. 143, 1990, pp. 455–460.

[16]O'Reilly, O., and Holmes, P. J., "Non-Linear, Non-Planar and Non-Periodic Vibrations of a String," *Journal of Sound and Vibration*, Vol. 153, 1992, pp. 413–435.

[17]Han, R. P. S., Luo, A. C. J., and Xiang, Y., "An Irrational Model for the Nonlinear Vibration of a String," *Journal of Nonlinear Mechanics* (submitted for publication).

[18]Han, R. P. S., and Luo, A. C. J., "Resonant Conditions for a Stretched Spinning Tether," 4th International Conf. on Tethers in Space, Washington, DC, 1995.

[19]Luo, A. C. J., Han, R. P. S., Modi, V. J., Misra, A. K., and Tyc, G., "Chaos in Undamped Stretched Spinning Tether," 15th CANCAM, Victoria, BC, Canada, 1995.

[20]Shaw, J., and Shaw, S. W., "Instabilities and Bifurcations in a Rotating Shaft," *Journal of Sound and Vibration*, Vol. 132, 1989, pp. 227–244.

[21]Luo, A. C. J., Han, R. P. S., and Xiang, Y., "Chaotic Analysis of Subharmonic Resonant Waves in Undamped and Damped Strings," *Journal of Hydrodynamics, Ser. B.* Vol. 7, 1995, pp. 92–104.

[22]Han, R. P. S., and Luo, A. C. J., "Analytical Predictions of Chaos in a Nonlinear Rod," *Journal of Nonlinear Dynamics* (submitted for publication).